Differential operator

Summary of differential operators in Cartesian coordinates (x, y, z); p, **u** and **τ** are scalar, vector and tensor fields, respectively.

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{u}\cdot\nabla = u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$$

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}$$

$$\nabla\cdot\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\nabla\times\mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}$$

$$\begin{aligned}\nabla\mathbf{u} = {} & \frac{\partial u_x}{\partial x}\mathbf{ii} + \frac{\partial u_y}{\partial x}\mathbf{ij} + \frac{\partial u_z}{\partial x}\mathbf{ik} + \frac{\partial u_x}{\partial y}\mathbf{ji} \\ & + \frac{\partial u_y}{\partial y}\mathbf{jj} + \frac{\partial u_z}{\partial y}\mathbf{jk} + \frac{\partial u_x}{\partial z}\mathbf{ki} + \frac{\partial u_y}{\partial z}\mathbf{kj} + \frac{\partial u_z}{\partial z}\mathbf{kk}\end{aligned}$$

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} = {} & \left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right)\mathbf{i} + \left(u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right)\mathbf{j} \\ & + \left(u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right)\mathbf{k}\end{aligned}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau} = {} & \left(\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right)\mathbf{i} + \left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}\right)\mathbf{j} \\ & + \left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z}\right)\mathbf{k}\end{aligned}$$

Differential operator

Summary of differential operators in cylindrical polar coordinates (r, θ, z); p, **u** and **τ** are scalar, vector and tensor fields, respectively.

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_\theta + \left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]\mathbf{e}_z$$

$$\begin{aligned}\nabla\mathbf{u} = {} & \frac{\partial u_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_r\mathbf{e}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_r\mathbf{e}_z + \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\mathbf{e}_\theta\mathbf{e}_r \\ & + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\mathbf{e}_\theta\mathbf{e}_z + \frac{\partial u_r}{\partial z}\mathbf{e}_z\mathbf{e}_r + \frac{\partial u_\theta}{\partial z}\mathbf{e}_z\mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\mathbf{e}_z\end{aligned}$$

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} = {} & \left[u_r\frac{\partial u_r}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + u_z\frac{\partial u_r}{\partial z}\right]\mathbf{e}_r \\ & + \left[u_r\frac{\partial u_\theta}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + u_z\frac{\partial u_\theta}{\partial z}\right]\mathbf{e}_\theta \\ & + \left[u_r\frac{\partial u_z}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right]\mathbf{e}_z\end{aligned}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau} = {} & \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r}\right]\mathbf{e}_r \\ & + \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} - \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right]\mathbf{e}_\theta \\ & + \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right]\mathbf{e}_z\end{aligned}$$

Differential operator

Summary of differential operators in spherical polar coordinates (r, θ, φ); p, **u** and **τ** are scalar, vector and tensor fields, respectively.

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\mathbf{u}\cdot\nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

$$\begin{aligned}\nabla\times\mathbf{u} = {} & [\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\phi \sin\theta) - \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}]\mathbf{e}_r + [\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(r u_\phi)]\mathbf{e}_\theta \\ & + [\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}]\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\nabla\mathbf{u} = {} & \frac{\partial u_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_r\mathbf{e}_\theta + \frac{\partial u_\phi}{\partial r}\mathbf{e}_r\mathbf{e}_\phi + \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\mathbf{e}_\theta\mathbf{e}_r \\ & + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\mathbf{e}_\theta\mathbf{e}_\phi + \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r}\right)\mathbf{e}_\phi\mathbf{e}_r \\ & + \left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{r}\cot\theta\right)\mathbf{e}_\phi\mathbf{e}_\theta + \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right)\mathbf{e}_\phi\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} = {} & [u_r\frac{\partial u_r}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r}\right)]\,\mathbf{e}_r \\ & + [u_r\frac{\partial u_\theta}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{r}\cot\theta\right)]\,\mathbf{e}_\theta \\ & + [u_r\frac{\partial u_\phi}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_\phi}{\partial \theta} + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right)]\,\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau} = {} & [\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta r}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi r}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}]\mathbf{e}_r \\ & + [\frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{\tau_{\theta r} - \tau_{r\theta} - \tau_{\phi\phi}\cot\theta}{r}]\mathbf{e}_\theta \\ & + [\frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\phi}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{\phi r} - \tau_{r\phi} - \tau_{\phi\theta}\cot\theta}{r}]\mathbf{e}_\phi\end{aligned}$$

Useful identities

Useful identities involving the nabla operator; f and g are scalar fields, and **u** and **v** are vector fields. It is assumed that all the partial derivatives involved are continuous.

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{u}\cdot\nabla)\,\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{u})$$

$$\nabla\cdot(f\mathbf{u}) = f\,\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla f$$

$$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v})$$

$$\nabla\cdot(\nabla\times\mathbf{u}) = 0$$

$$\nabla\times(f\mathbf{u}) = f\,\nabla\times\mathbf{u} + \nabla f\times\mathbf{u}$$

$$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\,\nabla\cdot\mathbf{v} - \mathbf{v}\,\nabla\cdot\mathbf{u} + (\mathbf{v}\cdot\nabla)\,\mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{v}$$

$$\nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u}$$

$$\nabla\times(\nabla f) = \mathbf{0}$$

$$\nabla(\mathbf{u}\cdot\mathbf{u}) = 2\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} + 2\mathbf{u}\times(\nabla\times\mathbf{u})$$

$$\nabla^2(fg) = f\,\nabla^2 g + g\,\nabla^2 f + 2\,\nabla f\cdot\nabla g$$

$$\nabla\cdot(\nabla f\times\nabla g) = 0$$

$$\nabla\cdot(f\,\nabla g - g\,\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$$

Differential operator: Physical significance

- Consider an infinitesimal volume ΔV bounded by a surface ΔS.
- The gradient of a scalar field f can be defined as

$$\nabla f \equiv \lim_{\Delta V \to 0} \frac{\int_{\Delta S} \mathbf{n}\, f\, dS}{\Delta V},$$

where **n** is the unit vector normal to the surface ΔS.

  - represents the *net vector flux* of the scalar quantity f at a point where the volume ΔV of surface ΔS collapses in the limit.

- The divergence of the velocity vector **u** can be defined as

$$\nabla \cdot \mathbf{u} \equiv \lim_{\Delta V \to 0} \frac{\int_{\Delta S} (\mathbf{n} \cdot \mathbf{u})\, dS}{\Delta V}$$

  - represents the *scalar flux* of the vector **u** at a point, which is equivalent to the *local rate of expansion.*

- The vorticity of **u** is defined as

$$\nabla \times \mathbf{u} \equiv \lim_{\Delta V \to 0} \frac{\int_{\Delta S} (\mathbf{n} \times \mathbf{u}) dS}{\Delta V}$$

  - represents *the vector net flux* of the scalar angular component at a point, which tends to rotate the fluid particle at the point where ΔV collapses.

The Substantial Derivative: Physical significance

- The time derivative represents the rate of change of a physical quantity experienced by an observer who can be either stationary or moving.
- In the case of fluid flow, a nonstationary observer may be moving exactly as a fluid particle or not.
- Hence, at least three different time derivatives can be defined in fluid mechanics and in transport phenomena.
- The classical example of fish concentration in a lake is illustrative of the similarities and differences between these time derivatives.
- Let c (x, y, t) be the fish concentration in a lake. For a stationary observer, say standing on a bridge and looking just at a spot of the lake beneath him, the time derivative is determined by the amount of fish arriving and leaving the spot of observation, i.e., the total change in concentration and thus the total time derivative, is identical to the *partial derivative*,

$$\frac{dc}{dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y}$$

  and is only a function of the local change of concentration. Imagine now the observer riding a boat which can move with relative velocity $\mathbf{u}^{Rel}$ *with respect to that of the water*. Hence, if $\mathbf{u}^{Boat}$ and $\mathbf{u}^{Water}$ are the velocities of the boat and the water, respectively, then

$$\mathbf{u}^{Rel} = \mathbf{u}^{Boat} + \mathbf{u}^{Water}$$

- The concentration now is a function not only of the time *t*, but also of the position of the boat $\mathbf{r}(x, y)$ too. The position of the boat is a function of time, and, in fact,

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}^{Rel}$$

The Substantial Derivative: Physical significance

and so $$\frac{dx}{dt} = u_x^{Rel} \quad \text{and} \quad \frac{dy}{dt} = u_y^{Rel}$$

- Thus, in this case, the *total time derivative* or the change experienced by the moving observer is,

$$\begin{aligned}\frac{d}{dt}[c(t,x,y)] &\equiv \left(\frac{\partial c}{\partial t}\right)_{x,y} + \left(\frac{\partial c}{\partial x}\right)_{t,y}\frac{dx}{dt} + \left(\frac{\partial c}{\partial y}\right)_{t,y}\frac{dy}{dt} = \\ &= \left(\frac{\partial c}{\partial t}\right)_{x,y} + u_x^{Rel}\left(\frac{\partial c}{\partial x}\right)_{t,y} + u_y^{Rel}\left(\frac{\partial c}{\partial y}\right)_{t,x}\end{aligned}$$

- Imagine now the observer turning off the engine of the boat so that $\mathbf{u}^{Boat} = \mathbf{0}$ and $\mathbf{u}^{Rel} = \mathbf{u}^{Water}$. Then,

$$\begin{aligned}\frac{d}{dt}[c(t,x,y)] &= \left(\frac{\partial c}{\partial t}\right)_{x,y} + \left(\frac{\partial c}{\partial x}\right)_{t,y}\frac{dx}{dt} + \left(\frac{\partial c}{\partial y}\right)_{t,x}\frac{dy}{dt} \\ &= \left(\frac{\partial c}{\partial t}\right)_{x,y} + u_x^{Water}\left(\frac{\partial c}{\partial x}\right)_{t,y} + u_y^{Water}\left(\frac{\partial c}{\partial y}\right)_{t,x} \\ &= \frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c\end{aligned}$$

- This derivative is called the *substantial derivative* and is denoted by *D/Dt* :

$$\frac{Dc}{Dt} \equiv \frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c$$

Note: Sometimes the terms *substantive*, *material* or *convective* are used for the *substantial* derivative.

The Substantial Derivative: Physical significance

- The substantial derivative expresses the total time change of a quantity, experienced by an observer following the motion of the liquid.

- It consists of a local change, $\partial c/\partial t$, which vanishes under steady conditions (i.e., same number of fish arrive and leave the spot of observation), and of a traveling (convective) change, $\mathbf{u} \cdot \nabla c$, which is zero for a stagnant liquid or uniform concentration.

- Thus, for a steady-state process,

$$\frac{Dc}{Dt} = \mathbf{u} \cdot \nabla c = u_1 \frac{\partial c}{\partial x_1} + u_2 \frac{\partial c}{\partial x_2} + u_3 \frac{\partial c}{\partial x_3}$$

- For stagnant liquid or uniform concentration,

$$\frac{Dc}{Dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} = \frac{dc}{dt}$$

Reference:
Viscous Fluid Flow, by Papanastasiou, Georgiou and Alexandrou, 2000.

The Substantial Derivative

Table. *The substantial derivative operator in various coordinate systems.*

Coordinate system	$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$
(x, y, z)	$\frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}$
(r, θ, z)	$\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$
(r, θ, ϕ)	$\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial}{\partial \phi}$