System Stability

- The most important concept in system response is stability.
- The term stability has many different definitions and uses, but the most common definition is related to equilibrium.
- A system in equilibrium will remain in the same state in the absence of external disturbances.
- A stable system will return to an equilibrium state if a “small” disturbance moves the system away from the initial state.
- An unstable system will not return to an equilibrium position, and frequently will move “far” from the initial state.
- Figure 5.1 illustrates three stability conditions with a simple ball and hill system.
- In each case an equilibrium position is easily identified—either the top of the hill or the bottom of the valley.
- In the unstable case, a small motion of the ball away from the equilibrium position will cause the ball to move “far” away, as it rolls down the hill.

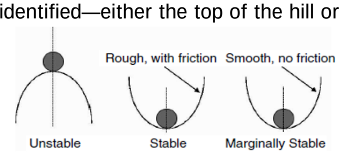

FIGURE 5.1 System stability.

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- In the stable case, a small movement of the ball away from the equilibrium position will eventually result in the ball returning, perhaps after a few oscillations.
- In the third case, the absence of friction causes the ball to oscillate continuously about the equilibrium position once a small movement has occurred. This special case is often known as marginal stability, since the system never quite returns to the equilibrium position.
- Most sensors and actuators are inherently stable. However, the addition of active control systems can cause a system of stable devices to exhibit overall unstable behavior.
- Careful analysis and testing is required to ensure that a mechatronic system acts in a stable manner.
- The complex response of stable dynamic systems is frequently approximated by much simpler systems. Understanding both first-order and second order system responses to either instantaneous (or step) changes in inputs or sinusoidal inputs will suffice for most situations.

System Stability: criteria

- A system is stable if *every* bounded input yields a bounded output.
  - this is called the bounded-input, bounded-output (BIBO) definition of stability.
- Physically, an unstable system whose natural response grows without bound can cause damage to the system, to adjacent property, or to human life.
  - Many times systems are designed with limited stops to prevent total runaway.
- Recall from our study of system poles that poles in the left half-plane (lhp) yield either pure exponential decay or damped sinusoidal natural responses.
  - These natural responses decay to zero as time approaches infinity.
  - Thus, if the closed-loop system poles are in the left half of the plane and hence have a negative real part, the system is stable.
  - That is, *stable systems have closed-loop transfer functions with poles only in the left half-plane.*
- Poles in the right half-plane (rhp) yield either pure exponentially increasing or exponentially increasing sinusoidal natural responses.
  - These natural responses approach infinity as time approaches infinity.
  - Thus, if the closed-loop system poles are in the right half of the s-plane and hence have a positive real part, the system is unstable.

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- – Thus, *unstable systems have closed loop transfer functions with at least one pole in the right half-plane and/or poles of multiplicity greater than 1 on the imaginary axis.*
- Finally, a system that has imaginary axis poles of multiplicity 1 yields pure sinusoidal oscillations as a natural response.
  - These responses neither increase nor decrease in amplitude.
  - Thus, *marginally stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity! and poles in the left half-plane.*

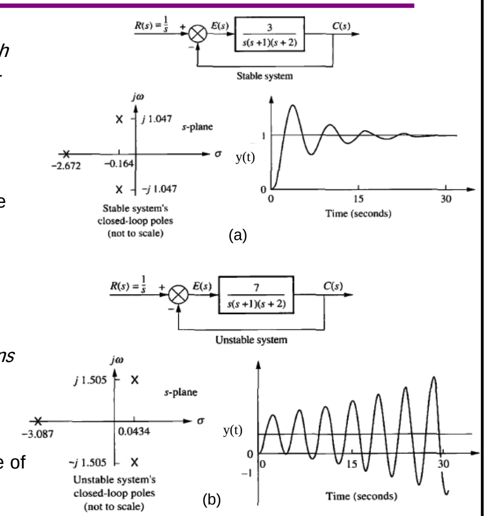


Figure Closed-loop poles and response of (a) stable system, (b) unstable system.

System Stability: Routh-Hurwitz Criterion

- a method that yields stability information without the need to solve for the closed-loop system poles.
- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the $j\omega$.
  - (Notice that we say *how many,* not *where.)*
- We can find the number of poles in each section of the s-plane, but we cannot find their coordinates.

- The method requires two steps:
  - (1) Generate a data table called a *Routh table*
  - (2) interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on *the* $j\omega$*-axis.*

Generating a Basic Routh Table

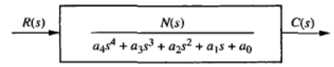


TABLE 1 Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

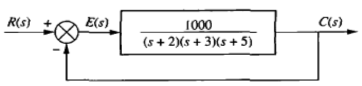
TABLE 2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- Routh-Hurwitz criterion states that *the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.*
- If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable.
- Thus, a system is stable if there are no sign changes in the first column of the Routh table.

Routh Table: Example 1

- Make the Routh table for the following system



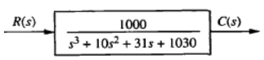


TABLE 3 Completed Routh table

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

- There are two sign changes in the first column.
  - The first sign change occurs from 1 in the s^2 row to -72 in the s^1 row.
  - The second occurs from -72 in the s^1 row to 103 in the s° row.
  - Thus, the system is unstable since two poles exist in the right half-plane.

Routh Table: Example 2: Stability via Epsilon Method: Case 1

- Two special cases can occur: (1) The Routh table sometimes will have a zero *only in the first column* of a row, or (2) the Routh table sometimes will have an *entire row* that consists of zeros. Let us examine the first case.
- Make the Routh table for the following system $T(s) = \dfrac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$

TABLE 4 Completed Routh table

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

TABLE 5 Determining signs in first column of a Routh table with zero as first element in a row

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

- If ε is chosen positive, Table 5 will show a sign change from the s^3 row to the s^2 row, and there will be another sign change from the s^2 row to the s^1 row. Hence, the system is unstable and has two poles in the right half-plane.
- Alternatively, if we choose ε negative. Table 5 will then show a sign change from the s^4 row to the s^3 row. Another sign change would occur from the s^3 row to the s^2 row. Our result would be exactly the same as that for a positive choice for *e.* Thus, the system is unstable, with two poles in the right half-plane.

Routh Table: Example 3: Entire Row is Zero: Case 2

- Determine the number of right-half-plane poles in the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Steps

- 1. Start by forming the Routh table for the denominator
- At the second row we multiply through by 1/7 for convenience.
- We stop at the third row, since the entire row consists of zeros, and use the following procedure.
- First we return to the row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients.
- The polynomial will start with the power of s in the label column and continue by skipping every other power of s. Thus, the polynomial formed for this example is Eq. (1).
- Next we differentiate the polynomial with respect to s and obtain
- we use the coefficients of Eq. (2) to replace the row of zeros

TABLE 6 Routh table

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

$$P(s) = s^4 + 6s^2 + 8 \quad (1)$$

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \quad (2)$$

Routh Table: Example 3: Entire Row is Zero: Case 2 (contd…)

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Steps

- Again, for convenience, the third row is multiplied by 1/4 after replacing the zeros.
- The remainder of the table is formed in a straightforward manner by following the standard form shown in Table 2.
- Table 6 shows that all entries in the first column are positive.
- Hence, there are no right-half-plane poles and the system is Stable.

TABLE 6 Routh table

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

$$P(s) = s^4 + 6s^2 + 8 \quad (1)$$

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \quad (2)$$