ME 261: Numerical Analysis

- 3.00 credit hours Prereq.: ME 163/ME 171

- **Course content**
  - Approximations and error types
  - Roots of polynomials and transcendental equations
  - Determinants and matrices
  - Solution of linear and non-linear algebraic equations
  - Eigenvalues and eigenvectors
  - Interpolation methods
  - Curve fitting
  - Numerical differentiation and integration
  - Solution of first-order differential equations
  - Solving equations by finite differences

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ME 261: Numerical Analysis

- **Course teachers**

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ME 261: Numerical Analysis

- **Text book**
  -Numerical Methods for Engineers
  by Steven C. Chapra and Raymond P. Canale
  Publishing company: Tata McGraw-Hill

- **Reference book**
  1. Applied Numerical Analysis
  by Curtis F. Gerald and Patrik O. Wheatley
  Publishing company: Pearson
  2. Numerical Analysis
  by Timothy Sauer
  Publishing company: Pearson

- **ACADEMIC OFFENCES**
  Students must write their assignments in their own words.
  If students take an idea, or a passage of text from book, journal, web etc, they must acknowledge this by proper referencing such as footnotes or citations.
  Plagiarism is a major academic offence.

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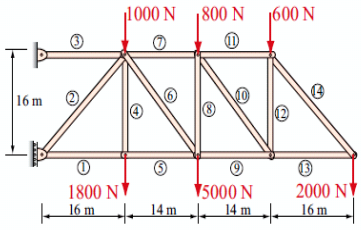
Why Numerical Analysis ?

- Example: computing the forces in a TRUSS



$$
\begin{aligned}
-F_1 + F_5 &= 0\\
-0.71071F_2 - F_3 + 0.6585F_6 + F_7 &= 0\\
-0.7071F_2 - F_4 - 0.7525F_6 &= 1000\\
F_4 &= 1800\\
-F_7 + 0.6585F_{10} + F_{11} &= 0\\
-F_8 - 0.7525F_{10} &= 800\\
-F_5 - 0.6585F_6 + F_9 &= 0\\
0.7525F_6 + F_8 &= 5000\\
-F_{11} + 0.7071F_{14} &= 0\\
-F_{12} - 0.7071F_{14} &= 600\\
-F_9 - 0.6585F_{10} + F_{13} &= 0\\
0.7525F_{10} + F_{12} &= 0\\
-F_{13} + 0.7071F_{14} &= 0\\
0.7071F_{14} &= 2000
\end{aligned}
$$

- 14 equations with 14 unknowns

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Floating Point Arithmetic

- fractional quantities are typically represented in computers using floating point format
- this approach is very much similar to scientific notation
- for example, *fixed point number* 17.542 is the same as the floating point number $.17542*10^2$ which is often displayed as .17542e2
- another example, -.004428 is same as $-.4428*10^{-2}$

General form of floating-point number in Computers:

$$\pm .d_1 d_2 d_3 \ldots . d_p * B^{e.}$$

where, d_j's are digits or bits with values from 0 to B-1
B=the number base that is used, e.g. 2, 10,16,8
P=number of significand bits (digits), that is, the precision.
e=an integer exponent, ranging from E_{min} to E_{max}

- $d_1 d_2 d_3 \ldots d_p$ constitute the fractional part of the number

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Floating Point Arithmetic

Normalization:

- the fractional digits are shifted and the exponents are adjusted so that d_1 is NONZERO

e.g.
$-.004428 \cong -0.4428*10^{-2}$ is the normalized form

- In base-2(binary) system normalization means *the first bit is always 1*

IEEE Standard for the floating-point numbers:

- consists of a set of binary representations of real numbers
- normalization gives the advantage that the first bit does not need to be stored
- Hence, nonzero binary floating point numbers can be expressed as,

$$\pm (1+ f)*2^e$$

where f =the mantissa (or the fraction)
e= exponent

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Floating Point Arithmetic

- e.g. the decimal number 9, which is 1001 in binary, would be stored as,

$$+1.001 * 2^3$$

although the original number has 4 significant digits, we only have to store the 3 fractional bits: .001

- There are three commonly used levels of precision for floating point numbers:
  - single precision (32 bits)
  - double precision (64 bits)
  - long double (80 bits)

The bits are divided as follows:

precision	sign	exponent	mantissa
single	1	8	23
double	1	11	52
long double	1	15	64

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Floating Point Arithmetic

- By default, MATLAB has adopted IEEE double precision format

Figure: The manner in which a floating point number is stored in IEEE double precision format

Note: because of normalization, 53 bits can be stored in mantissa.

- Sign bit is 0 for positive numbers and 1 for negative numbers

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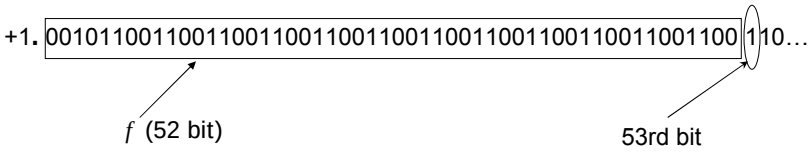
Floating Point Arithmetic

The decimal number $9.4 = (1001.\overline{0110})_2 = (1.00\overline{10110})_2 * 2^3$

Normalized

It will be stored in exponent



f (52 bit)

53rd bit

How do we fit the infinite binary number representing 9.4 in a finite number of bits?

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We must truncate the number in some way, and in doing so we necessarily make a small error

We can use two methods to truncate

Chopping:

- Simply through away the bits that fall off the end

  -that is, those beyond the 52nd bit

- This method is simple, but it is biased in that it always moves the result toward zero

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Rounding:

- In base 10, numbers are customarily rounded up if the next digit is 5 or higher, and rounded down otherwise
- In binary, this corresponds to rounding up if the bit is 1
- Specifically, the important bit in the double precision format is the 53rd bit

IEEE Rounding to Nearest Rule:

if bit 53 is 0,then truncate after the 52nd bit (round down)
if bit 53 is 1, add 1 to 52nd bit (round up)
exception: if the bits 53rd and beyond are like 10000000...,
then add 1 to 52nd bit if and only if 52nd bit is 1.

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For the number (9.4) discussed previously, the 53rd bit in mantissa is 1

+1. 0010110011001100110011001100110011001100110011001100 110…

53rd bit

So, the Rounding to the Nearest Rule says to round up, i.e. to add 1 to bit 52

Therefore the mantissa of floating point number that represents 9.4 is

+1. 0010110011001100110011001100110011001100110011001101

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Now let's see how much error occurs due to this rounding of the mantissa

We have discarded the bits after 53rd, so error associated with discarding (Ed) is

$$= .\overline{110} * 2^{-52} * 2^{3} \quad = .\overline{0110} * 2^{-51} * 2^{3} \quad = .4 * 2^{-48}$$

We have added 1 in the 52nd bit, so error associated with roundup (Er) is

$$2^{-52} * 2^{3} = 2^{-49}$$

Therefore, fl(9.4)=9.4+Er-Ed

$$= 9.4 + 2^{-49} - 0.4 * 2^{-48} = 9.4 + (1 - 0.8) * 2^{-49}$$

$$= 9.4 + 0.2 * 2^{-49}$$

Roundoff error

In other words, a computer using double precision representation and the Rounding to the nearest Rule makes as error of $0.2*2^{-49}$

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Machine Epsilon ε_{mach}

52 bits of mantissa

The double precision number 1 is

+1. 0000000000000000000000000000000000000000000000000000

The next floating point number greater than 1 is

+1. 0000000000000000000000000000000000000000000000000001

$\Rightarrow 1+2^{-52}$

So, the number machine Epsilon, denoted by ε_{mach}, is the distance between 1 and the smallest floating point number greater than 1.

For the IEEE double precision floating point standard,

$$\varepsilon_{mach} = 2^{-52} = 2.2204 * 10^{-16}$$

It is a quantitative measure of the machine's ability to perform computations accurately, depending upon the precision in use.

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Machine Epsilon ε_{mach}

- Double precision numbers can be computed reliably

up to machine epsilon.

- MATLAB has a library function **eps** to display ε_{mach}

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Absolute and relative error

Let X_{comp} be a computed version of the exact quantity X_{exact}

Absolute error $= |X_{comp} - X_{exact}|$ Relative error $= \dfrac{|X_{comp} - X_{exact}|}{|X_{exact}|}$

- absolute error does not account the order of magnitude of the value under examination.
- Relative error can be expressed in %
- For example, we want to measure the length of bridge and a rivet.
- We measured the length of bridge is 9999 cm and the length of the rivet is 9 cm.
- The exact values are 10000 and 10 cm, respectively.
- The absolute error for both cases is 1 cm.
- The percentage relative errors are 0.01% and 10%.
- That means, measurement of the length of the bridge is quite accurate, but the measurement of the rivet has significant error.

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Arithmetic accuracy in Computers

- *Roundoff errors* arise because digital computers cannot represent some quantities exactly.
- Aside from the limitations of a computer's number system, the actual arithmetic manipulation involving this numbers can also result in roundoff error.
- To understand how this occurs, let's look at how the computer performs simple addition and subtraction.
- Because of their familiarity, normalized base-10 numbers will be employed to illustrate the effect of roundoff errors on simple arithmetic manipulations.
- Other number bases would behave in a similar fashion.
- To simplify the discussion, we assume a hypothetical computer with a 3-digit mantissa, and a 1-digit exponent, i.e. B=10, p=3, $-9 \le e \le 9$.

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Arithmetic accuracy in Computers

- When two floating-point numbers are added /or subtracted,
  - the numbers are first expressed so that they have the same exponents.

examples,

Addition:

- if we want to add 1.371 + 0.02693
- the computer would express the numbers as

$$\begin{array}{rl} & 0.1371 \times 10^{1} \\ + & 0.002693 \times 10^{1} \\ \hline & 0.139793 \times 10^{1} \end{array}$$

Align the decimal points

- Because this hypothetical computer only carries 3-digit mantissa
  if rounded, → 0.140×10^{1}
  if chopped, → 0.139×10^{1}

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Arithmetic accuracy in Computers

Subtraction:

- compute 76.4 – 76.3 on a 3-decimal-digit computer.
- The computer would express the numbers as

$$\begin{array}{rll} & 0.764 & * 10^2 \\ - & 0.763 & * 10^2 \\ \hline & 0.001 & * 10^2 \end{array}$$

3 significant digits

- After normalization, $0.100 * 10^0$

Two non-significant zeros are appended

Subtractive cancelation/loss of significant:

- subtraction of two nearly equal numbers.
- a major source of error in floating-point operations.

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Subtractive cancellation/loss of significant

- subtraction of two nearly equal numbers
- a major source of error in floating-point operations

Example-1

Example-2

Lesson

- It is important to find ways to avoid subtracting nearly equal numbers in calculations

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Arithmetic accuracy in Computers

- Roundoff error can happen in several circumstances other than just storing numbers - for example:
  - *Large computations* - if a process performs a large number of computations, roundoff errors may build up to become significant
    - Example
  - *Adding a Large and a Small Number* - Since the small number's mantissa is shifted to the right to be the same scale as the large number, digits are lost
  - *Smearing* - Smearing occurs whenever the individual terms in a summation are larger than the summation itself.
    - $(x+10^{-20})-x = 10^{-20}$ mathematically, but

      x=1; $(x+1e^{-20})-x$ gives a 0 in MATLAB!

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Adding a Large and a Small Number

- Suppose we want to add a small number 0.001 to a large number 4000 using a hypothetical computer with the 4-digit mantissa and the 1-digit exponent

4000	0.400	$* 10^4$
0.001	0.000 000 1	$* 10^4$
	0.400 000 1	$*10^4$

- The final result will be $0.4000 *10^4$ (since we have 4-digit mantissa)
- It seems that we have not performed any addition !!!
- This type of error may occur in the computation of an infinite series

Example

- One way to mitigate this type of error is to sum the series in reverse order
- In this way each new term will be of comparable magnitude to the accumulated sum

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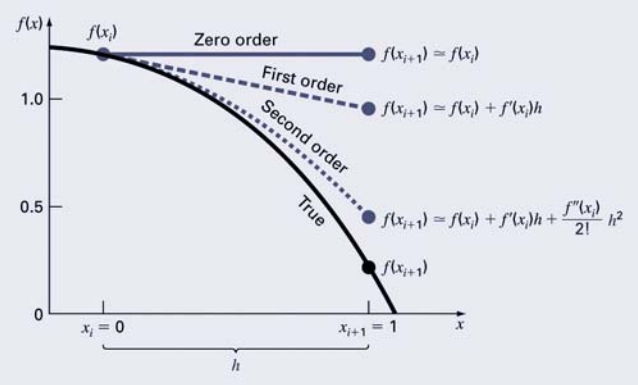
Truncation Error

Truncation errors are those that result from using an approximation in place of an exact mathematical procedure.

Example: The Taylor Series

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n$$



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Truncation Error

R_n =O(h^{n+1}), meaning:

- The more terms are used, the smaller the error, and
- The smaller the spacing, the smaller the error for a given number of terms.

Example 2: Numerical differentiation

- The first order Taylor series can be used to calculate approximations to derivatives:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

If h is decreased (↓) truncation error decreases(↓)

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Error Propagation

- Functions of a single variable
  - Derivation using Taylor series for a single variable
  - Example
- Functions of more than one variable
  - Derivation using multivariable version of Taylor series
  - Example

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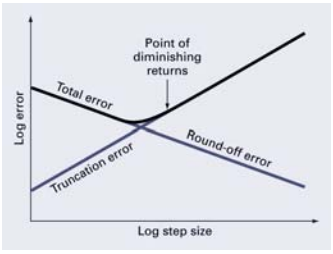
Total Numerical Error

- The *total numerical error* is the summation of the truncation and roundoff errors.
- The truncation error generally *increases* as the step size increases,
- To minimize the roundoff error, increase the number of precision (bits in mantissa) in computers
- Round off error increases due to subtractive cancellation
  - Or due to an increase in the number of computations in an analysis.
- A decrease in stepsize can lead to subtractive cancellation, or to an increase in computations, $\Rightarrow$ the round off errors are increased.



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Other Errors

- Blunders - errors caused by malfunctions of the computer or human imperfection.
- Model errors - errors resulting from incomplete mathematical models.
- Data uncertainty - errors resulting from the inaccuracy and/or imprecision of the measurement of the data.

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